# Pointwise Convergence

Note Title 3/8/2010

Here is a basic theorem, which is a combination of results of Dirichlet and Dini.

<u>Theorem</u>: Let $f$ be $2\pi$-periodic and suppose

$$\frac{f(x_0+t)-f(x_0)}{t} = g(t) \in \mathcal{R}[-\pi,\pi]. \quad \text{Let}$$

$$s_N f(x_0) = \sum_{-N}^{N} c_n e^{inx_0}, \text{ where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

Then $\lim_{N\to\infty} s_N f(x_0) = f(x_0)$.

<u>proof</u>: $f(x_0+t) - f(x_0) = t\cdot g(t) \in \mathcal{R}[-\pi,\pi]$

So $f(x_0+t) \in \mathcal{R}[-\pi,\pi]$ which means $f \in \mathcal{R}[-\pi,\pi]$.

So $c_n$ is defined. Let $D_N(t) = \frac{1}{2\pi}\sum_{-N}^{N} e^{int} = \frac{1}{2\pi}\left(\frac{e^{-iNt} - e^{i(N+1)t}}{1-e^{it}}\right)$.

$$s_N f(x_0) = \int_{-\pi}^{\pi} f(t)\left(\frac{1}{2\pi}\sum_{N}^{N} e^{in(x_0-t)}\right) dt$$

$$= \int_{-\pi}^{\pi} f(t)\, D_N(x_0-t)\,dt = \int_{-\pi}^{\pi} f(x_0+t)\, D_N(t)\,dt.$$

$$s_N f(x_0) - f(x_0) = \int_{-\pi}^{\pi} \left[f(x_0+t) - f(x_0)\right] D_N(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(x_0+t)-f(x_0)}{t}\cdot\frac{t}{(1-e^{it})}\cdot\left[e^{-iNt} - e^{i(N+1)t}\right]$$

$$\frac{f(x_0+t)-f(x_0)}{t}\cdot\frac{t}{1-e^{it}} = g(t)\cdot\frac{t}{1-e^{it}} \in R[-\pi,\pi],$$

so

$$s_N f(x_0) - f(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(x_0+t)-f(x_0)}{t}\cdot\frac{t}{(1-e^{it})}\cdot\left[e^{-iNt}-e^{i(N+1)t}\right] \to 0$$

as $N\to\infty$ by the Riemann-Lebesgue lemma.

QED.

Cor: If $f(x)=g(x)$ on an open set around $x_0$ and if $f, g \in R[-\pi,\pi]$, then

$$\lim_{N\to\infty}\left[s_N f(x_0) - s_N g(x_0)\right] = 0.$$

(The behaviour of the Fourier series at $x_0$ of $f$ is determined entirely by the behaviour of $f$ in an arbitrarily small neighborhood of $x_0$. It's values elsewhere don't matter.)

proof: Apply the theorem to $f-g$.